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Bidimensional optical solitons in a quadratic medium

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Abstract. The modulation evolution of a short localized optical pulse in a crystal belonging to one of the classes $\bar{4}2m$, $\bar{4}3m$, $3m$, $6mm$, and with a non-vanishing second-order nonlinearity, is considered. In $(2+1)$ dimensions, the partial differential system accounting for it can be reduced to the completely integrable Davey–Stewartson system, if some conditions are satisfied. The first integrability condition represents a balance between the third-order Kerr effect and the cascaded second-order nonlinearities, while the second condition is an equilibrium between the dispersion and the kinetic factor of the electro-optic–optical rectification wave interaction. For anomalous dispersion, the obtained Davey–Stewartson system is of the type I, that admits localized soliton solutions. Lump solution, algebraically decaying in all directions, exist in any case satisfying the above conditions.

1. Introduction

Multidimensional optical solitons are at the present time the subject matter of intensive research and growing interest, for both fundamental and applicative reasons. The cascading of second-order optical nonlinearities is one of the phenomena on which most hopes are founded for realizing such effects. In $(1+1)$ dimensions, the soliton theory rests on the theory of the equations that are completely integrable through the inverse scattering transform (IST) method. The most common and the most important of these equations is the nonlinear Schrödinger (NLS) equation. In more than $(1+1)$ dimensions, completely integrable equations are rare. However, there exists an integrable $(2+1)$-dimensional generalization of the NLS equation: the so-called Davey–Stewartson (DS I and DS II) system. Localized soliton solutions have been found for this system (see the survey and references in section 3), but to my knowledge, these mathematical solutions have never been experimentally observed. Further, their occurrence in a given physical frame has never been theoretically predicted. It is well known that partial differential systems with a form analogous to the Davey–Stewartson one can describe the evolution of a short localized pulse modulation in a quadratic optical medium [1]. The wave is stabilized by an interaction with some d.c. field, through optical rectification and electro-optic effect. However, the derivation of the model was achieved in a rather heuristic way. A recent paper [2] by Leblond presents for the first time the derivation of such a model in $(3+1)$ dimensions, taking into account the tensorial structure of the susceptibilities, for some particular symmetry classes of crystals ($\bar{4}2m$ or $\bar{4}3m$ and $3m$ or $6mm$). This enables us to give in this paper the conditions under which localized solitons may arise in a thin sheet of some second-order nonlinear material belonging to one of these classes.

This paper is organized as follows, first we recall the results of [2], second we summarize the principal properties of the Davey–Stewartson system and then the reduction of the

$(3+1)$-dimensional model to the Davey–Stewartson system, together with the integrability conditions, is presented in section 4. Section 4.1 for the $\bar{4}2m$ and $\bar{4}3m$ symmetry classes, section 4.2 for the $3m$ class, and section 4.3 for the $6mm$ class which behaves as the $3m$ one, with some slight changes. The results are then simplified using the 'complete symmetry' property of the susceptibilities, and commented on in section 5. A conclusion summarizes these results and their interpretation.

2. A (3 + 1)-dimensional model that describes wave modulation in crystals

In a recent paper [2], we derived a $(3+1)$-dimensional system of partial differential equations, that describes the evolution of a short localized optical pulse in a bulk sample of some crystal with a nonzero $\chi^{(2)}$ susceptibility, far from the phase matching. Aside from the self-modulation of the wave through both cascading and Kerr effect, it shows the interaction between the wave and some d.c. field, or rather some solitary electromagnetic wave, varying with the same spatial and temporal scales as the wave modulation. This interaction, due to optical rectification and electro-optic effect, is essential, because it can stabilize the pulse: in particular we show in this paper that, under certain (restrictive) conditions, the $(2+1)$-dimensional evolution of the pulse obeys the Davey–Stewartson system, which is completely integrable by the IST method, thus the pulse has all properties of solitons, regarding stability, robustness, and interactions.

Further, our study takes into account the tensorial structure of the linear and nonlinear susceptibilities. This is crucial for the above-mentioned results. Thus we consider successively two particular structures of the nonlinear susceptibilities, corresponding to the $\bar{4}2m$ and $3m$ symmetry classes of crystals. These classes are physically the most important ones: the $\bar{4}2m$ class contains potassium dihydrogen phosphate (KDP) and all analogous materials, but also contains others. All results obtained for the $\bar{4}2m$ class are also valid for the $\bar{4}3m$ class, to which for example, GaAs belongs. The $3m$ class is not less important while the lithium niobate, which is nowadays the matter of extensive research, belongs to it, together with other materials. The results valid for this class are also valid for the $6mm$ class, which contains, for example, CdSe.

We use slow variables (τ, ξ, η, ζ) (see [2] for their precise definition). ξ and η are the transverse space variables, τ is the variable describing the shape of the pulse, and ζ the variable describing the evolution of this shape during the propagation of the pulse. Let us call $\mathcal{E}^x$ and $\mathcal{E}^y$ respectively the x and y components of the wavefield, and $\mathcal{E}_0^x$, $\mathcal{E}_0^y$ the x and y components of the d.c. field or solitary wave. The model system obtained in [2] reads, in the case of the $\bar{4}2m$ symmetry class:

$$[2\mathrm{i}k\partial_\zeta + \beta\partial_\xi^2 + \partial_\eta^2 - kk''\partial_\tau^2]\mathcal{E}^x + (\beta-1)\partial_\xi\partial_\eta\mathcal{E}^y$$
$$= D_1\mathcal{E}^x|\mathcal{E}^x|^2 + D_2\mathcal{E}^x|\mathcal{E}^y|^2 + D_3(\mathcal{E}^y)^2\mathcal{E}^{x,*} + E\Phi\mathcal{E}^y \tag{1}$$
$$[2\mathrm{i}k\partial_\zeta + \partial_\xi^2 + \beta\partial_\eta^2 - kk''\partial_\tau^2]\mathcal{E}^y + (\beta-1)\partial_\xi\partial_\eta\mathcal{E}^x$$
$$= D_1\mathcal{E}^y|\mathcal{E}^y|^2 + D_2\mathcal{E}^y|\mathcal{E}^x|^2 + D_3(\mathcal{E}^x)^2\mathcal{E}^{y,*} + E\Phi\mathcal{E}^x \tag{2}$$
$$[\alpha(\partial_\xi^2 + \partial_\eta^2) + \rho\partial_\tau^2]\Phi = \lambda(\partial_\xi^2 + \partial_\eta^2)(\mathcal{E}^x\mathcal{E}^{y,*} + \mathcal{E}^{x,*}\mathcal{E}^y). \tag{3}$$

The function Φ is some combination of the zero harmonic components $\mathcal{E}_0^x$ and $\mathcal{E}_0^y$, given by

$$\Phi = \int^\tau (\partial_\xi\mathcal{E}_0^x + \partial_\eta\mathcal{E}_0^y). \tag{4}$$

The constants α and β give an account of the anisotropy, respectively at the frequencies zero and ω:

$$\alpha = \frac{n_o^2(0)}{n_e^2(0)} \qquad \beta = \frac{n_o^2}{n_e^2} \tag{5}$$

ρ measures the difference between the group velocity v of the fast oscillating wave and the speed $\frac{c}{n_o(0)}$ of the solitary wave yielded by the rectified field.

$$\rho = \frac{1}{v^2} - \frac{n_o^2(0)}{c^2} = \frac{(n_o + \omega n_o')^2 - n_o^2(0)}{c^2} \tag{6}$$

Depending on the sign of ρ, equation (3) is either a Poisson equation or a wave equation. An analogous phenomenon has been studied in ferromagnetic media, for relatively larger input powers [3]. In the case where $\rho < 0$, the wave equation can describe the backscattering of some shock wave, in a way that can be compared with the Tcherenkov effect, with longer wavelengths, while the scattered solitary wave behaves smoothly when ρ is positive. This ρ factor also appears in the denominator of the nonlinear constants of the NLS equations derived from the same $(3+1)$-dimensional system. If it takes small enough values, it can lead to large self-phase modulation, even if the corresponding $\chi^{(2)}$ component is small. This has already been noticed by Newell [1].

The interaction constants read as follows

$$D_1 = \frac{-3\omega^2}{c^2}\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega) \tag{7}$$

$$\begin{aligned} D_2 = {} & \frac{4\omega^2}{c^2}\frac{1}{n_e^2(2\omega)}\hat{\chi}^{(2)}_{xzy}(2\omega,-\omega)\hat{\chi}^{(2)}_{zxy}(\omega,\omega) + \frac{4\omega^2}{c^2}\frac{1}{n_e^2(0)}\hat{\chi}^{(2)}_{xzy}(0,\omega)\hat{\chi}^{(2)}_{zxy}(\omega,-\omega) \\ & -\frac{3\omega^2}{c^2}(\hat{\chi}^{(3)}_{xxyy}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xyxy}(\omega,\omega,-\omega)) \end{aligned} \tag{8}$$

$$D_3 = \frac{4\omega^2}{c^2}\frac{1}{n_e^2(0)}\hat{\chi}^{(2)}_{xzy}(0,\omega)\hat{\chi}^{(2)}_{zxy}(\omega,-\omega) - \frac{3\omega^2}{c^2}\hat{\chi}^{(3)}_{xyyx}(\omega,\omega,-\omega) \tag{9}$$

$$E = \frac{-2\omega^2}{c^2}\frac{vn_o^2(0)}{n_e^2(0)}\hat{\chi}^{(2)}_{xzy}(0,\omega) \tag{10}$$

$$\lambda = \frac{2}{vn_e^2(0)}\hat{\chi}^{(2)}_{xzy}(\omega,-\omega). \tag{11}$$

λ gives account for the interaction $\omega + (-\omega) \longrightarrow 0$, which produces the zero harmonic or mean-value term $\mathcal{E}_0^x$, $\mathcal{E}_0^y$ (the rectified field). The constant E accounts for the interaction $\omega + 0 \longrightarrow \omega$ of the latter with the fundamental, which is the electro-optic effect, and, in addition to the third-order Kerr effect, $D_3 = \frac{-1}{\alpha}E\lambda - \frac{3\omega^2}{c^2}\hat{\chi}^{(3)}_{xyyx}(\omega,\omega,-\omega)$ accounts for the cascading of the two preceding interactions, while $(D_2 - D_3)$ describes the cascaded interactions $\omega + \omega \longrightarrow 2\omega$ and $2\omega + (-\omega) \longrightarrow \omega$.

For the $3m$ symmetry class, the obtained system reads as follows

$$\begin{aligned} & [2\mathrm{i}k\partial_\zeta + \beta\partial_\xi^2 + \partial_\eta^2 - kk''\partial_\tau^2]\mathcal{E}^x + (\beta-1)\partial_\eta\partial_\xi\mathcal{E}^y \\ & \quad = D_1\mathcal{E}^x|\mathcal{E}^x|^2 + D_2\mathcal{E}^x|\mathcal{E}^y|^2 + D_3(\mathcal{E}^y)^2\mathcal{E}^{x,*} + F_1(\mathcal{E}_0^x\mathcal{E}^y + \mathcal{E}_0^y\mathcal{E}^x) \\ & \quad\quad + F_2\mathcal{E}^x\int^\tau(\partial_\xi\mathcal{E}_0^x + \partial_\eta\mathcal{E}_0^y) \\ & \quad\quad + F_3\mathcal{E}^x\int^\tau[\partial_\xi(\mathcal{E}^x\mathcal{E}^{y,*} + \mathcal{E}^y\mathcal{E}^{x,*}) + \partial_\eta(|\mathcal{E}^x|^2 - |\mathcal{E}^y|^2)] \end{aligned} \tag{12}$$

$$
\begin{aligned}
&[2\mathrm{i}k\partial_\zeta + \partial_\xi^2 + \beta\partial_\eta^2 - kk''\partial_\tau^2]\mathcal{E}^y + (\beta-1)\partial_\eta\partial_\xi\mathcal{E}^x \\
&\qquad = D_1\mathcal{E}^y|\mathcal{E}^y|^2 + D_2\mathcal{E}^y|\mathcal{E}^x|^2 + D_3(\mathcal{E}^x)^2\mathcal{E}^{y,*} + F_1(\mathcal{E}_0^x\mathcal{E}^x - \mathcal{E}_0^y\mathcal{E}^y) \\
&\qquad\quad + F_2\mathcal{E}^y\int^\tau(\partial_\xi\mathcal{E}_0^x + \partial_\eta\mathcal{E}_0^y) \\
&\qquad\quad + F_3\mathcal{E}^y\int^\tau[\partial_\xi(\mathcal{E}^x\mathcal{E}^{y,*} + \mathcal{E}^y\mathcal{E}^{x,*}) + \partial_\eta(|\mathcal{E}^x|^2 - |\mathcal{E}^y|^2)]
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&[\alpha\partial_\xi^2 + \partial_\eta^2 + \rho\partial_\tau^2]\mathcal{E}_0^x + (\alpha-1)\partial_\eta\partial_\xi\mathcal{E}_0^y = (\lambda_1\partial_\xi^2 + \lambda_3\partial_\tau^2)(\mathcal{E}^x\mathcal{E}^{y,*} + \mathcal{E}^y\mathcal{E}^{x,*}) \\
&\qquad\quad + \lambda_1\partial_\xi\partial_\eta(|\mathcal{E}^x|^2 - |\mathcal{E}^y|^2) + \lambda_2\partial_\xi\partial_\tau(|\mathcal{E}^x|^2 + |\mathcal{E}^y|^2)
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&[\partial_\xi^2 + \alpha\partial_\eta^2 + \rho\partial_\tau^2]\mathcal{E}_0^y + (\alpha-1)\partial_\eta\partial_\xi\mathcal{E}_0^x = (\lambda_1\partial_\eta^2 + \lambda_3\partial_\tau^2)(|\mathcal{E}^x|^2 - |\mathcal{E}^y|^2) \\
&\qquad\quad + \lambda_1\partial_\xi\partial_\eta(\mathcal{E}^x\mathcal{E}^{y,*} + \mathcal{E}^y\mathcal{E}^{x,*}) + \lambda_2\partial_\eta\partial_\tau(|\mathcal{E}^x|^2 + |\mathcal{E}^y|^2).
\end{aligned}
\tag{15}
$$

The constants are given in appendix A, formulae (48)–(59).

Both systems (1)–(3) and (12)–(15), which describe the evolution of a three-dimensional pulse in a bulk sample of some material belonging to the $\bar{4}2m$ class, or to the $3m$ class respectively, are expected to yield the NLS equation:

$$
\mathrm{i}A\partial_\zeta f + B\partial_X^2 f + Cf|f|^2 = 0 \tag{16}
$$

when reduced to $(1+1)$ dimensions, and a single polarization. However, this reduction, extensively studied in [2], is not as simple as one would think at first glance. Indeed, it can be achieved for a few special polarizations only, and for a particular choice of the variable X. Generally speaking, X can either be the longitudinal 'time' variable τ, or some transverse variable $r\xi + s\eta$, r and s being real constants. In the latter case, known as 'spatial', the reduction to NLS is only possible if r and s take some special values, which describe some particular choice of the modulation direction. From the physical point of view, r and s determine the orientation of the planar waveguide relative to the crystallographic axes: this orientation can only take a few special values. Neither is the polarization free: elliptic polarizations are forbidden, as are circular polarizations in the anisotropic spatial case, while the linear polarizations must make some fixed angle with the waveguide plane, determined by the crystal symmetry. Furthermore, the coefficients of the NLS equation (16) depend strongly on the considered case.

The systems (1)–(3) or (12)–(15) are thus $(3+1)$-dimensional generalizations of the NLS equation, in the sense that they can be reduced to the latter, at least under some conditions, despite the fact that conditions are very restrictive. However, it is very unlikely that these systems, such as the NLS equation (16), possess the property of being completely integrable by the IST method, even for very particular values of the coefficients. Indeed, completely integrable systems are extremely rare in $(3+1)$ dimensions. One has been discovered by Leon [4], but it is radically different from the present one. It is essential to notice that these systems are not the 'classical' $(3+1)$-dimensional generalization of the NLS, the so-called three-dimensional NLS equation, which is obtained from equation (16) by replacing the ∂_X^2 derivative by a three-dimensional Laplacian operator. The three-dimensional NLS equation is known to only have unstable solutions [5], but this result is by no means valid for our systems. The latter involve at least one degree of freedom more: the d.c. field.

3. The Davey–Stewartson equations

A system exists which is somehow analogous in its form to the previous ones, that describes the evolution of a wave amplitude in $(2+1)$ dimensions, and that is, like the NLS equation,

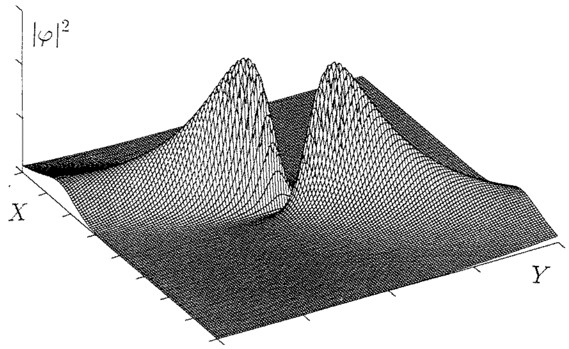

Figure 1. A lump solution of the Davey–Stewartson system. Plot of the square modulus of the fundamental amplitude φ versus X and $Y \propto \tau$, for some given value of the propagation variable ζ.

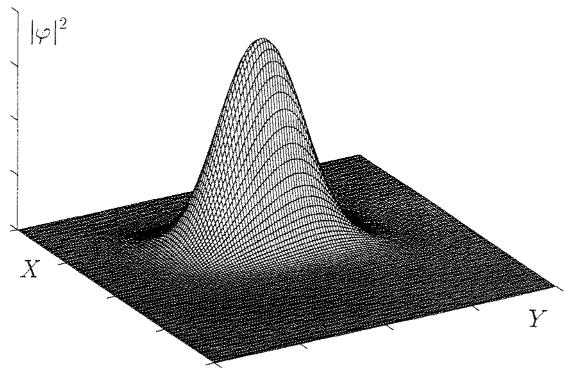

Figure 2. A localized soliton solution of the Davey–Stewartson system. The notations are the same as in figure 1.

completely integrable by the IST method: the so-called Davey–Stewartson system. This system was first derived by Davey and Stewartson in the frame of surface water waves [6]. In the 1970s it was shown that this system is completely integrable through the IST method, and the Hirota bilinear form was also found. An N-soliton solution was obtained, but, although the solitons can propagate in different directions, each of them is quasi-one-dimensional. Lump, and N-lump solutions, which are solutions algebraically decaying in all directions, were found in 1978 by Satsuma and Ablowitz, using a generalization of the Hirota method [7] (figure 1). True solitons, i.e. solutions exponentially decaying in all directions, have been found by Boiti *et al*, using the IST method [8–10] (figure 2). Hietarinta and Hirota found a multiple localized soliton solution, which they call 'N^2-dromion', using the bilinear formalism [11]. It must be noticed that these localized solitons or dromions assume nonvanishing boundary conditions at infinity for the auxiliary field (denoted by Ψ in equations (17), (18) below). In the optical frame, this field describes some solitary microwave interacting with the optical pulse. The physical interpretation of the nonvanishing boundary conditions for Ψ, and the question as to whether they may be interpreted as a possibility to control the optical pulse through the microwave field, are left for further investigation. An interesting review of the known properties of the system is given in [12].

The general form of the Davey–Stewartson system reads

$$\mathrm{i}\partial_\zeta\varphi + \varepsilon_1\partial_X^2\varphi + \partial_Y^2\varphi + \varepsilon_2\varphi|\varphi|^2 + \nu\varphi\Psi = 0 \tag{17}$$

$$\partial_X^2\Psi + \mu\partial_Y^2\Psi = \partial_X^2|\varphi|^2. \tag{18}$$

The unknown functions φ and Ψ take respectively complex and real values. ν, μ are real coefficients, and ε_1, $\varepsilon_2 = \pm 1$. Every system with a form analogous to these equations (17), (18), but with arbitrary coefficients, can be reduced to this form by means of a linear change of variables. However, unlike the NLS equation, system (17) and (18) are not completely integrable for any value of the coefficients, but only if the following conditions are satisfied:

$$\varepsilon_1\mu = \varepsilon_2\nu + 1 \tag{19}$$

and

$$\varepsilon_1\mu = -1. \tag{20}$$

If condition (19) only is satisfied, the system still possesses a Hirota bilinear form, and admits exact one-solitary wave solutions. Condition (20) is required for the existence of the two-solitary waves solution. It ensures also the complete integrability.

These integrability conditions are checked in the following way: considering the equations solved in the quoted papers, conditions (19) and (20) are clearly sufficient. To show that they are necessary requires some computation: first we write explicitly the Hirota bilinear form for equations (17) and (18), as [7], and we see that condition (19) is necessary. We then compute the one-soliton solution through the Hirota method, and notice that the computation is valid, even if condition (20) is not satisfied. We then compute the two-solitons solution, and condition (20) arises as a solvability condition. The computation is lengthy but without difficulty: it follows strictly the known Hirota procedure [13].

Four cases exist for the signs ε_1 and ε_2. The case $\varepsilon_1 = 1$ is referred to as DS I, and the case $\varepsilon_1 = -1$ as DS II, for both values of ε_2. The existence of localized solutions depends on the case. The solutions obtained initially were called soliton solutions, because they have the same mathematical characteristics as the soliton solutions of the NLS equation, with regard to the IST method, or to the Hirota bilinear method, but they are not localized in space. They are in fact quasi-one-dimensional. Such solutions exist for the four sign cases, which are all completely integrable. The so-called one-soliton solution is still valid if condition (20) is not satisfied.

However, solutions decaying in all directions do not always exist. The DS II equations with $\varepsilon_2 = -1$, does not admit such solutions. It is regrettable, since it is the case found by Davey and Stewartson in water theory, and it is still, to my knowledge, the only sign case that has been derived in a physical frame. The lump solutions [7], that decay algebraically in all directions, exist when $\varepsilon_1\varepsilon_2 = -1$, i.e. partly for DS I, and partly for DS II. The exponentially decaying solutions, the true multidimensional solitons of Boiti *et al* exist only for DS I (but for both $(\varepsilon_1, \varepsilon_2) = (1, 1)$ and $(1, -1)$). The multidromion of [11] is defined in the case $(\varepsilon_1, \varepsilon_2) = (1, -1)$, for which lumps and 'true' solitons exist.

4. Reduction of the $(3 + 1)$-dimensional system to the Davey–Stewartson equations

4.1. Case of the $\bar{4}2m$ symmetry class

Let us look now for reductions of the systems (1)–(3) to the integrable DS equations. First we seek for the ability of propagating a single polarization. We set

$$\mathcal{E}^x = f\cos\theta \qquad \mathcal{E}^y = f\sin\theta \tag{21}$$

θ is then the angle between the polarization direction and the x-axis, and f is some amplitude to be determined. The spatial (relative to the variables ξ and η) partial differential operator in the l.h.s. of equations (1) and (2) must be the same. This necessitates the restriction to one spatial dimension defined by a variable X, which is

$$X = \xi \cos\theta + \eta \sin\theta \tag{22}$$

or

$$X = -\xi \sin\theta + \eta \cos\theta. \tag{23}$$

The first case corresponds to a modulation direction (that is the direction of the planar waveguide) parallel to the polarization direction; and the second to a modulation perpendicular to the polarization. We will write $(X \parallel f)$ or $(X \perp f)$ to label these two cases. The spatial differential operator reduces to $B\partial_X^2$, with $B = \beta$ (if $X \parallel f$), or $B = 1$ (if $X \perp f$).

Then we compare the coefficients of $f|f|^2$ in equations (1) and (2), and see that they coincide only if the condition $\cos^2\theta = \sin^2\theta$ is satisfied. Thus $\theta = \pm\frac{\pi}{4}$. Equations (1)–(3) are then reduced to the following ones:

$$[\alpha\partial_X^2 + \rho\partial_\tau^2]\Phi' = \lambda\partial_X^2|f|^2 \tag{24}$$

$$[2\mathrm{i}k\partial_\zeta + B\partial_X^2 - kk''\partial_\tau^2]f = \frac{D_1 + D_2 + D_3}{2} f|f|^2 + Ef\Phi' \tag{25}$$

($\Phi' = \Phi$ if $\theta = \frac{\pi}{4}$, $-\Phi$ if $\theta = \frac{-\pi}{4}$).

Equations (24), (25) can be reduced to the DS system (17), (18), by the following linear change of variables:

$$t = \zeta\frac{\varepsilon_1 B}{2k} \qquad X = X \qquad Y = \tau\sqrt{\left|\frac{B}{-kk'}\right|} \tag{26}$$

$$\varphi = f\sqrt{\frac{|D_1 + D_2 + D_3|}{2B}} \qquad \Psi = \Phi'\frac{\alpha}{\lambda}\frac{|D_1 + D_2 + D_3|}{2B}. \tag{27}$$

The constants are:

$$\begin{aligned}
\varepsilon_1 &= \mathrm{sgn}(-kk'') \\
\varepsilon_2 &= -\varepsilon_1\,\mathrm{sgn}(D_1 + D_2 + D_3) \\
\mu &= \frac{\varepsilon_1\rho B}{-kk''\alpha} \\
\nu &= \varepsilon_2\frac{2\lambda}{\alpha}\frac{E}{D_1 + D_2 + D_3}.
\end{aligned} \tag{28}$$

The integrability condition (20) then reads

$$\frac{(n_o + \omega n_o')^2 - n_o^2(0)}{n_o\omega(n_o''\omega + 2n_o')}\frac{n_e^2(0)}{n_o^2(0)}B = 1. \tag{29}$$

Recall that

$$B = \frac{n_o^2}{n_e^2}\ (\text{if } X \parallel f) \qquad \text{or} \qquad 1\ (\text{if } X \perp f). \tag{30}$$

If this condition is satisfied, the other integrability condition (19) reads,

$$\begin{aligned}
&3[\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xxyy}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xyxy}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xyyx}(\omega,\omega,-\omega)] \\
&\quad = \frac{4}{n_e^2(0)}\hat{\chi}^{(2)}_{xzy}(0,\omega)\hat{\chi}^{(2)}_{zxy}(\omega,-\omega) + \frac{4}{n_e^2(2\omega)}\hat{\chi}^{(2)}_{xzy}(2\omega,-\omega)\hat{\chi}^{(2)}_{zxy}(\omega,\omega).
\end{aligned} \tag{31}$$

Written under this form, the integrability condition can appear as an equilibrium between the third-order nonlinearity and the cascaded second-order one. If condition (29) is not satisfied, the less restrictive condition (20) for the existence of the bilinear form and of the one-soliton solution (quasi-one-dimensional) writes,

$$\varepsilon_1\mu - 1 = \varepsilon_2\nu = \frac{-1}{1+K} \tag{32}$$

where $-\varepsilon_1\mu$ is the l.h.s. of equation (29) and K is defined as:

$$K = \frac{n_e^2(0)}{2\hat{\chi}^{(2)}_{xzy}(0,\omega)\hat{\chi}^{(2)}_{zxy}(\omega,-\omega)}\left[\frac{1}{n_e^2(2\omega)}\hat{\chi}^{(2)}_{xzy}(2\omega,-\omega)\hat{\chi}^{(2)}_{zxy}(\omega,\omega) - \frac{3}{4}(\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega)\right.$$
$$\left. + \hat{\chi}^{(3)}_{xxyy}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xyxy}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xyyx}(\omega,\omega,-\omega))\right]. \tag{33}$$

When conditions (29) and (31) are satisfied, the signs ε_1 and ε_2 are important for the existence of localized solitons. They exist for the DS I equation, that is, when $\varepsilon_1 = \mathrm{sgn}(-kk'') = +1$. That is, when $k'' < 0$: for anomalous dispersion. The sign $\varepsilon_1\varepsilon_2$ determines the existence of lump solutions: using equation (31), we find that

$$-\varepsilon_1\varepsilon_2 = \mathrm{sgn}(\hat{\chi}^{(2)}_{xzy}(0,\omega)\hat{\chi}^{(2)}_{zxy}(\omega,-\omega)). \tag{34}$$

Thus if $\hat{\chi}^{(2)}_{xzy}(0,\omega)\hat{\chi}^{(2)}_{zxy}(\omega,-\omega)$ is positive, the lump solutions exist.

4.2. Case of the 3m symmetry class.

The reduction of equations (12)–(15) to the Davey–Stewartson system (17) and (18) is not as easy as the previous one. It is described with detail in appendix B. It is first seen that the direction of the modulation (the variable X, physically the plane of the waveguide) must be either parallel or perpendicular to the polarization (modulated by the function f). (As an abbreviation we write, as above, $(X \parallel f)$ or $(X \perp f)$.) Second, it is seen that the angle θ, that describes the direction of the polarization, can take only the values $\frac{-\pi}{2}$, $\frac{\pi}{6}$, $\frac{5\pi}{6}$ (case $(X \parallel f)$) or 0, $\frac{2\pi}{3}$, $\frac{-2\pi}{3}$ (case $(X \perp f)$). In both cases, the solutions form an equilateral triangle, coherent with the trigonal $3m$ symmetry of the crystal.

The integral terms are removed without additional hypothesis, by introducing an adequate auxiliary field Φ', defined by equation (70). A remarkable simplification (equation (73)) allows this removal. However, the auxiliary equation (74) takes the required form (18) only if the following condition is satisfied:

$$\hat{\chi}^{(2)}_{yyy}(0,\omega)\hat{\chi}^{(2)}_{zxx}(\omega,-\omega) + \hat{\chi}^{(2)}_{xzx}(0,\omega)\hat{\chi}^{(2)}_{yyy}(\omega,-\omega) = 0. \tag{35}$$

A linear change of the variables (given in appendix B, equations (79)–(83)) then reduces the equations to the form of (17) and (18). The integrability conditions (19), (20) can then be written. Condition (20) is the same as for the $\bar{4}2m$ symmetry class (equation (29)), while the second condition ($\varepsilon_2\nu = -2$) takes a form somehow analogous to (31), but more complicated:

$$\frac{1}{n_o^2(0)}\frac{(n_o+\omega n_o')^2 - 2n_o^2(0)}{(n_o+\omega n_o')^2 - n_o^2(0)}\hat{\chi}^{(2)}_{yyy}(0,\omega)\hat{\chi}^{(2)}_{yyy}(\omega,-\omega) + \frac{1}{n_e^2(0)}\hat{\chi}^{(2)}_{xzx}(0,\omega)\hat{\chi}^{(2)}_{zxx}(\omega,-\omega)$$
$$+\frac{1}{n_o^2(2\omega) - n_o^2}\hat{\chi}^{(2)}_{yyy}(2\omega,-\omega)\hat{\chi}^{(2)}_{yyy}(\omega,\omega) + \frac{1}{n_e^2(2\omega)}\hat{\chi}^{(2)}_{xzx}(2\omega,-\omega)\hat{\chi}^{(2)}_{zxx}(\omega,\omega)$$
$$= \tfrac{3}{2}\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega). \tag{36}$$

The sign ε_1 determines whether the obtained system is DS I or DS II, thus whether it admits localized soliton solutions or not. This sign is $\varepsilon_1 = \mathrm{sgn}(-kk'')$, as for the $\bar{4}2m$ class. The existence of lump solutions is determined by the sign $-\varepsilon_1\varepsilon_2$, which reads, after some algebra using the above integrability conditions:

$$-\varepsilon_1\varepsilon_2 = \mathrm{sgn}\left(\frac{(n_o+\omega n_o')^2}{n_o^2(0)-(n_o+\omega n_o')^2}\hat{\chi}^{(2)}_{yyy}(0,\omega)\hat{\chi}^{(2)}_{yyy}(\omega,-\omega) + \frac{n_o^2(0)}{n_e^2(0)}\hat{\chi}^{(2)}_{xzx}(0,\omega)\hat{\chi}^{(2)}_{zxx}(\omega,-\omega)\right). \tag{37}$$

If this quantity is positive, the equations admit lump solutions, algebraically decaying in all directions.

4.3. *Case of the 6mm symmetry class*

The symmetry class $6mm$, to which many $\chi^{(2)}$-materials belong, has a $\chi^{(2)}$-structure of the same form as the $3m$ class [14], but with $\hat{\chi}^{(2)}_{yyy} = 0$, and the same $\chi^{(3)}$-structure, also with some zero coefficients. The derivations and discussion above concerning the $3m$ class is valid for these materials, but is more simplified. Condition (35), which allows the reduction of the evolution equations to the system of (17), (18), is always satisfied, and while the integrability condition (29) is unchanged, the other one becomes

$$\frac{1}{n_e^2(0)}\hat{\chi}^{(2)}_{xzx}(0,\omega)\hat{\chi}^{(2)}_{zxx}(\omega,-\omega) + \frac{1}{n_e^2(2\omega)}\hat{\chi}^{(2)}_{xzx}(2\omega,-\omega)\hat{\chi}^{(2)}_{zxx}(\omega,\omega) = \tfrac{3}{2}\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega) \tag{38}$$

which is very close to the condition (31) for the integrability in the case of the $\bar{4}2m$ symmetry class.

5. Discussion of the integrability conditions

There are two integrability conditions: the first (29), concerns the dispersion relation and its derivatives: the velocities, the second (31), (36) or (38) concerns the susceptibilities. The condition (29) comes from (19), which states that the coefficient of ∂_Y^2 (that is, of ∂_τ^2) is the same in both equations, with a sign change. It can be rewritten as

$$\alpha kk'' = \rho B = \left(\frac{1}{v^2} - \frac{n_0^2(0)}{c^2}\right)B. \tag{39}$$

Leaving aside the terms α and $B = 1$ or β, that describe a distortion due to anisotropy, this represents an equilibrium between the dispersion coefficient kk'' and the ρ coefficient. ρ measures the difference between the group velocity v of the wave and the velocity $\frac{c}{n_o(0)}$ of the backscattered solitary wave, due to optical rectification. The dispersion coefficient kk'' measures similarly the difference between the group velocity of the wave, and those of its side-bands. In both cases, these velocity differences give account for the strength of the diffusive effect. Equation (39) thus represents an equilibrium between the kinetic terms of diffusive effects for optical rectification on the one hand, and dispersion on the other.

The other condition can be written in a simpler form by using the so-called complete symmetry property [14]. For the $\bar{4}2m$ and $\bar{4}3m$ classes, the condition (31) writes,

$$3[\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xxyy}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xyxy}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xyyx}(\omega,\omega,-\omega)] = \frac{4}{n_e^2(0)}(\hat{\chi}^{(2)}_{xzy}(0,\omega))^2 + \frac{4}{n_e^2(2\omega)}(\hat{\chi}^{(2)}_{xzy}(2\omega,-\omega))^2. \tag{40}$$

For the $3m$ class, condition (36) reads,

$$\frac{1}{n_o^2(0)}\frac{(n_o+\omega n_o')^2-2n_o^2(0)}{(n_o+\omega n_o')^2-n_o^2(0)}(\hat{\chi}^{(2)}_{yyy}(\omega,-\omega))^2+\frac{1}{n_e^2(0)}(\hat{\chi}^{(2)}_{xzx}(\omega,-\omega))^2 + \frac{1}{n_o^2(2\omega)-n_o^2}(\hat{\chi}^{(2)}_{yyy}(\omega,\omega))^2+\frac{1}{n_e^2(2\omega)}(\hat{\chi}^{(2)}_{xzx}(\omega,\omega))^2=\frac{3}{2}\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega) \tag{41}$$

and for the $6mm$ class, condition (38) writes,

$$\frac{1}{n_e^2(0)}(\hat{\chi}^{(2)}_{xzx}(\omega,-\omega))^2+\frac{1}{n_e^2(2\omega)}(\hat{\chi}^{(2)}_{xzx}(\omega,\omega))^2=\frac{3}{2}\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega). \tag{42}$$

In every case, this describes an equilibrium between the third-order Kerr coefficient and the cascaded second-order nonlinearities, through both second and zero harmonics. The complete symmetry property allows us to solve condition (35), that is necessary for the reduction of the $(3+1)$-dimensional system to the Davey–Stewartson equations, in the case of the $3m$ and $6mm$ classes. This condition reduces to

$$\hat{\chi}^{(2)}_{yyy}(\omega,-\omega)\cdot\hat{\chi}^{(2)}_{xzx}(\omega,-\omega)=0. \tag{43}$$

One of the factors must be zero. If $\hat{\chi}^{(2)}_{xzx}(\omega,-\omega)=0$, the integration condition (36) writes,

$$\frac{1}{n_o^2(0)}\frac{(n_o+\omega n_o')^2-2n_o^2(0)}{(n_o+\omega n_o')^2-n_o^2(0)}(\hat{\chi}^{(2)}_{yyy}(\omega,-\omega))^2+\frac{1}{n_o^2(2\omega)-n_o^2}(\hat{\chi}^{(2)}_{yyy}(\omega,\omega))^2 + \frac{1}{n_e^2(2\omega)}(\hat{\chi}^{(2)}_{xzx}(\omega,\omega))^2=\frac{3}{2}\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega). \tag{44}$$

Else if $\hat{\chi}^{(2)}_{yyy}(\omega,-\omega)=0$,

$$\frac{1}{n_e^2(0)}(\hat{\chi}^{(2)}_{xzx}(\omega,-\omega))^2+\frac{1}{n_o^2(2\omega)-n_o^2}(\hat{\chi}^{(2)}_{yyy}(\omega,\omega))^2+\frac{1}{n_e^2(2\omega)}(\hat{\chi}^{(2)}_{xzx}(\omega,\omega))^2 =\frac{3}{2}\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega). \tag{45}$$

The above interpretation for these conditions is more obvious after this simplification. Note that a case where $\hat{\chi}^{(2)}_{yyy}$ is identically equal to zero is the case of the $6mm$ class, and as above-mentioned, condition (35) is always satisfied for this class.

The last but not least observation is the sign condition for the existence of localized solution. The condition for the existence of lump solutions is $\varepsilon_1\varepsilon_2=-1$, and this sign is given by equations (34) or (37), depending on the symmetry class. Using the complete symmetry property, we find that $\varepsilon_1\varepsilon_2$ is the sign of the opposite of some square. Thus the lump solution always exist. This is valid for the classes $\bar{4}2m$, $\bar{4}3m$, $6mm$, and $3m$ when condition (35) is solved by $\hat{\chi}^{(2)}_{yyy}(\omega,-\omega)=0$. In the case of the $3m$ class, when this condition is solved by $\hat{\chi}^{(2)}_{zxx}(\omega,-\omega)=0$, we see that,

$$-\varepsilon_1\varepsilon_2=\mathrm{sgn}(-\rho).$$

Thus lump solutions exist when $\rho<0$, this condition appears also for the existence of exponentially decaying solitons, discussed hereafter.

On the other hand, the DS I equation, that is the case $\varepsilon_1=+1$, admits localized soliton solutions, exponentially decaying in all directions (and not only algebraically decaying as the lump solutions). It has been seen above that this occurs for anomalous dispersion ($k''<0$). However, through condition (39), k'' has the same sign as ρ, thus $\varepsilon_1=+1$ when,

$$v>\frac{c}{n_o(0).}$$

In this sign case, the emission of the slowly varying wave would be, for large enough input power, a backscattered shock wave.

6. Conclusion

The partial differential system describing the evolution of a three-dimensional pulse in a bulk sample of some $\chi^{(2)}$-material belonging to one of the symmetry classes $\bar{4}2m$, $\bar{4}3m$, $3m$ or $6mm$ has been reduced to the $(2+1)$-dimensional Davey–Stewartson system. The integrability conditions have been written down and discussed: the interaction with the d.c. field can stabilize the waves when mainly two conditions are satisfied. The first condition states that the kinetic factor for the efficiency of the backscattering of the d.c. wave balances the dispersion; the second one that the sum of the squared second-order nonlinear coefficients, corresponding to second-harmonic generation and optical rectification, must equilibrate the third order Kerr coefficient. Further, there are two different sign conditions that ensure the existence of localized solutions. The first one ensures the existence of exponentially decaying solutions: proper bidimensional solitons. It is satisfied for anomalous dispersion, which implies, because of the 'kinetic' integrability condition, that the backscattered rectified wave is a shock wave. The second is the condition for the existence of algebraically decaying solutions called lump: except for one special symmetry case, this condition is always satisfied.

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Appendix A

In this appendix, we give the expressions of the coefficients of the system of (12)–(15) of evolution equations for the $3m$ class. They read as follows:

$$\kappa_2 = \frac{-\hat{\chi}^{(2)}_{yyy}(\omega,\omega)}{n_o^2 - n_o^2(2\omega)} \tag{46}$$

$$\kappa_3 = \frac{-\hat{\chi}^{(2)}_{zxx}(\omega,\omega)}{n_e^2(2\omega)} \tag{47}$$

$$D_1 = K_1 + K_2 + K_3 - \frac{3\omega^2}{c^2}\hat{\chi}^{(3)}_{xxxx}(\omega,\omega,-\omega) \tag{48}$$

$$D_2 = 2K_2 + K_3 - \frac{3\omega^2}{c^2}(\hat{\chi}^{(3)}_{xxyy}(\omega,\omega,-\omega) + \hat{\chi}^{(3)}_{xyxy}(\omega,\omega,-\omega)) \tag{49}$$

$$D_3 = K_1 - K_2 - \frac{3\omega^2}{c^2}\hat{\chi}^{(3)}_{xyyx}(\omega,\omega,-\omega) \tag{50}$$

$$K_1 = \frac{-2\omega^2}{c^2}\hat{\chi}^{(2)}_{xzx}(2\omega,-\omega)\kappa_3 \tag{51}$$

$$K_2 = \frac{2\omega^2}{c^2}\hat{\chi}^{(2)}_{yyy}(2\omega,-\omega)\kappa_2 \tag{52}$$

$$K_1 = \frac{4\omega^2}{c^2 n_e^2(0)} \hat{\chi}^{(2)}_{xzx}(0,\omega)\hat{\chi}^{(2)}_{zxx}(\omega,-\omega) \tag{53}$$

$$F_1 = \frac{2\omega^2}{c^2} \hat{\chi}^{(2)}_{yyy}(0,\omega) \tag{54}$$

$$F_2 = \frac{-2\omega^2}{c^2} \frac{v n_o^2(0)}{n_e^2(0)} \hat{\chi}^{(2)}_{xzx}(0,\omega) \tag{55}$$

$$F_3 = \frac{4\omega^2}{c^2} \frac{v}{n_e^2(0)} \hat{\chi}^{(2)}_{xzx}(0,\omega)\hat{\chi}^{(2)}_{yyy}(\omega,-\omega) \tag{56}$$

$$\lambda_1 = \frac{2}{n_e^2(0)} \hat{\chi}^{(2)}_{yyy}(\omega,-\omega) \tag{57}$$

$$\lambda_2 = \frac{2}{v n_e^2(0)} \hat{\chi}^{(2)}_{zxx}(\omega,-\omega) \tag{58}$$

$$\lambda_3 = \frac{-2}{c^2} \hat{\chi}^{(2)}_{yyy}(\omega,-\omega) = \frac{-n_e^2(0)}{c^2}\lambda_1. \tag{59}$$

Appendix B. Reduction of the evolution equations to the Davey–Stewartson system for the $3m$ class

First, we want to find the possibility of propagating a single polarization. We set

$$\mathcal{E}^x = af \qquad \mathcal{E}^y = bf \tag{60}$$

and, as for the $\bar{4}2m$ class, the spatial differential operator in the l.h.s. of equations (12) and (13) coincide only if we restrict the problem to one spatial variable $X = r\xi + s\eta$, and if the condition

$$r^2 - s^2 + \left(\frac{b}{a} - \frac{a}{b}\right) rs = 0 \tag{61}$$

is satisfied. Solving equation (61), we find that $\frac{r}{s} = \frac{a}{b}$ or $\frac{-b}{a}$. Thus $\frac{a}{b}$ is real; we take $a = \cos\theta$, $b = \sin\theta$, and we have either

$$X = \xi\cos\theta + \eta\sin\theta \;\; (\text{if } X \parallel f)$$

or (62)

$$X = -\xi\sin\theta + \eta\cos\theta \;\; (\text{if } X \perp f).$$

This condition is clear, considering equations (12), (13), if we neglect the rotation of the axes measured by θ. Propagating $\mathcal{E}^x$ alone is possible only if the term that depends on $\mathcal{E}^y$ disappears. The differential operator in this term is factorized into $\partial_\xi\partial_\eta$, it cancels clearly only if $\mathcal{E}^y$ does not depend either on η, or on ξ.

If condition (62) is satisfied, the l.h.s. of the two equations are identical, and then the r.h.s. must also coincide. Due to the following symmetry properties of the $\chi^{(3)}$-tensor [14]:

$$\chi^{(3)}_{xxxx} = \chi^{(3)}_{yyyy} = \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyyx} + \chi^{(3)}_{xyxy}$$

we have: $D_2 + D_3 = D_1$, and thus the terms proportional to $f|f|^2$ are the same, and so are the integral terms. The following terms remain to be considered in the r.h.s. of equations (12) and (13), respectively:

$$F_1(\mathcal{E}_0^y + \tan\theta\,\mathcal{E}_0^x) f$$

and

$$F_1(\cot\theta\mathcal{E}_0^x - \mathcal{E}_0^y)f.$$

These terms coincide only if $\mathcal{E}_0^x$ and $\mathcal{E}_0^y$ are proportional, with a ratio equal to $\tan 2\theta$. We thus set

$$\begin{aligned}\mathcal{E}_0^x &= \sin 2\theta\Phi\\ \mathcal{E}_0^y &= \cos 2\theta\Phi.\end{aligned} \tag{63}$$

Then equations (12) and (13) become identical.

Now we turn to the auxiliary equations (14) and (15). We write them, using (63), and see that their r.h.s. coincide if the condition

$$\begin{aligned}r &= \sin 2\theta\\ s &= \cos 2\theta\end{aligned} \tag{64}$$

is satisfied. Then equations (14) and (15) reduce to

$$[\alpha\partial_X^2 + \rho\partial_\tau^2]\Phi = [\lambda_1\partial_X^2 + \lambda_2\partial_X\partial_\tau + \lambda_3\partial_\tau^2]|f|^2 \tag{65}$$

and the coefficients of equations (12) and (13) are computed to yield:

$$[2\mathrm{i}k\partial_\zeta + B\partial_X^2 - kk''\partial_\tau^2]f = D_1 f|f|^2 + F_1 f\Phi + F_2 f\int^\tau \partial_X\Phi + F_3 f\int^\tau \partial_X|f|^2. \tag{66}$$

We recall that:

$$B = \beta \text{ (if } X \parallel f) \qquad \text{or} \qquad 1 \text{ (if } X \perp f). \tag{67}$$

Before we pursue the reduction of equations (65) and (66) to the Davey–Stewartson system of (17) and (18), we look for the values of the angle θ that satisfy the previous conditions. For the modulation perpendicular to the polarization $(X \perp f)$, the conditions are

$$r = \sin 2\theta = -\sin\theta \qquad \text{and} \qquad s = \cos 2\theta = \cos\theta. \tag{68}$$

There are three solutions: $\theta = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$. For the the modulation parallel to the polarization $(X \parallel f)$, we must have

$$r = \sin 2\theta = \cos\theta \qquad \text{and} \qquad s = \cos 2\theta = \sin\theta. \tag{69}$$

There are also three solutions: $\theta = \frac{-\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

Then we set:

$$\Phi' = F_1\Phi + F_2\int^\tau \partial_X\Phi + F_3\int^\tau \partial_X|f|^2. \tag{70}$$

Equation (66) reduces to:

$$[2\mathrm{i}k\partial_\zeta + B\partial_X^2 - kk''\partial_\tau^2]f = D_1 f|f|^2 + f\Phi'. \tag{71}$$

Using equation (65), we compute the quantity $[\alpha\partial_X^2+\rho\partial_\tau^2]\partial_\tau\Phi'$, and this yields the following evolution equation for Φ':

$$[\alpha\partial_X^2 + \rho\partial_\tau^2]\partial_\tau\Phi' = [G_0\partial_X^3 + G_1\partial_X^2\partial_\tau + G_2\partial_X\partial_\tau^2 + G_3\partial_\tau^3]|f|^2 \tag{72}$$

with:

$$G_0 = \alpha F_3 + \lambda_1 F_2 = 0. \tag{73}$$

Thus we can integrate equation (72) once with respect to τ, and obtain

$$[\alpha\partial_X^2 + \rho\partial_\tau^2]\Phi' = [G_1\partial_X^2 + G_2\partial_X\partial_\tau + G_3\partial_\tau^2]|f|^2. \tag{74}$$

Equations (70) and (74) can now be reduced to the Davey–Stewartson equations (17), (18) if $G_2 = 0$. Computing $G_2 = F_1\lambda_2 + F_2\lambda_3 + F_3\rho$, we obtain condition (35).

The constants in equation (74) read,

$$G_1 = F_1\lambda_1 + F_2\lambda_2 \tag{75}$$

$$G_1 = \frac{4\omega^2}{c^2 n_e^2(0)} \left[\hat{\chi}^{(2)}_{yyy}(0,\omega)\hat{\chi}^{(2)}_{yyy}(\omega,-\omega) - \frac{n_o^2(0)}{n_e^2(0)}\hat{\chi}^{(2)}_{xzx}(0,\omega)\hat{\chi}^{(2)}_{zxx}(\omega,-\omega)\right] \tag{76}$$

$$G_3 = F_1\lambda_3 \tag{77}$$

$$G_3 = \frac{-4\omega^2}{c^4}\hat{\chi}^{(2)}_{yyy}(0,\omega)\hat{\chi}^{(2)}_{yyy}(\omega,-\omega). \tag{78}$$

We put:

$$t = \zeta\frac{\varepsilon_1 B}{2k} \qquad X = X \qquad Y = \tau\sqrt{\left|\frac{B}{-kk'}\right|} \tag{79}$$

$$\varphi = f\sqrt{\frac{1}{B}\left|\frac{G_3}{\rho} + D_1\right|} \tag{80}$$

$$\Phi' = \frac{\Psi}{q} + \frac{|\varphi|^2}{r} \tag{81}$$

with

$$q = \varepsilon_1\varepsilon_2\frac{\alpha}{B}\cdot\frac{G_3 + \rho D_1}{\alpha G_3 - \rho G_1} \tag{82}$$

$$r = \frac{-\varepsilon_1\varepsilon_2}{B}\cdot\frac{G_3 + \rho D_1}{G_3}. \tag{83}$$

The signs $\varepsilon_1\varepsilon_2$ are defined by:

$$\varepsilon_1 = \mathrm{sgn}(-kk'') \tag{84}$$

as for the $\bar{4}2m$ class and

$$\varepsilon_2 = -\varepsilon_1\,\mathrm{sgn}\left(\frac{G_3}{\rho} + D_1\right). \tag{85}$$

The introduction of a term proportional to $|\varphi|^2$ in formula (81) allows the term $G_3\partial_\tau^2|f|^2$ in equation (74) to vanish.

The system then reduces to the Davey–Stewartson equations (17) and (18), with the following value of the constants:

$$\mu = \frac{\varepsilon_1\rho B}{-kk''\alpha} \tag{86}$$

$$\nu = \frac{\varepsilon_2}{\alpha}\frac{\rho G_1 - \alpha G_3}{G_3 + \rho D_1}. \tag{87}$$

This allows us to write the integrability conditions (19) and (20), that yield conditions (29) and (36).

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